First-order ordinary differential equations: homogeneous equations

We say that a first-order ODE (meaning the highest derivative is $y'(x)$) is a homogeneous equation if it has the form:

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous functions of the same degree.

At this point, it is useful to review some properties of homogeneous functions, especially the following:

Reduction to $n-1$ variables:

$$\frac{f(x,y)}{x^n} = \varphi\left(\frac{y}{x}\right) = \varphi(v) \quad \text{for } x \neq 0, \quad y = vx$$

or

$$\frac{f(x,y)}{y^n} = \delta\left(\frac{x}{y}\right) = \delta(w) \quad \text{for } y \neq 0, \quad w = x/y$$

To solve this type of equation, we proceed with the following steps:

i. Divide the entire ODE by x^n (where n is the homogeneity degree of $P(x,y)$ and $Q(x,y)$).

ii. Make a convenient substitution: $v = \frac{y}{x}$.

Procedure

Step 1: Division by x^n

Divide the ODE by x^n (where n is the degree of homogeneity for $P(x,y)$ and $Q(x,y)$).

Step 2: Substitution

Substitute $v = \frac{y}{x}$. Then:

$$y = v\,x \quad \Rightarrow \quad dy = d(vx) \quad \Rightarrow \quad dy = v\,dx + x\,dv$$

This substitution transforms a homogeneous equation into a separable variables form. At this stage, operate on the equation to separate v and x.

Example

$$y^2 + (x^2 - xy)\,y' = 0$$

- This is a first-order, nonlinear equation with variable coefficients.
- Since $y' = \frac{dy}{dx}$, we can rewrite the equation as:

$$y^2 + (x^2 - xy)\frac{dy}{dx} = 0 \quad \Rightarrow \quad y^2\,dx + (x^2 - xy)\,dy = 0$$

- In this equation: $P(x,y) = y^2$ and $Q(x,y) = x^2 - xy$, both are homogeneous functions of degree 2.

Following the solution steps:

i. Divide the entire ODE by x^2:

$$\left(\frac{y}{x}\right)^2 dx + \left(1 - \frac{y}{x}\right) dy = 0$$

ii. Substitute conveniently: $v = \frac{y}{x} \Rightarrow dy = v\,dx + x\,dv$.

Substituting into the equation, we get:

$$(v)^2\,dx + (1 - v)(v\,dx + x\,dv) = 0$$

$$v^2\,dx + v\,dx + x\,dv - v^2\,dx - vx\,dv = 0$$

$$v\,dx + (x - vx)\,dv = 0$$

$$v\,dx + (1 - v)x\,dv = 0$$

Now, we apply the **Separation of Variables** method:

$$v\,dx = (v - 1)x\,dv$$

$$\frac{dx}{x} = \left(1 - \frac{1}{v}\right) dv$$

We can integrate both sides:

$$\int \frac{1}{x}\,dx = \int \left(1 - \frac{1}{v}\right) dv$$

This gives:

$$\ln x = v - \ln v + C$$

Next, substitute back to return to $y(x)$:

$$\ln x + \ln\left(\frac{y}{x}\right) = \frac{y}{x} + C \Rightarrow \ln\left(x \cdot \frac{y}{x}\right) = \ln y = \frac{y}{x} + C$$

Thus, the general solution is:

$$y = \frac{y}{x + C} = k\,e^x$$